Revisiting Boole Equation in the Quantum Context

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Abstract

In this work we try to clarify the fundamental relationship between *bits* and *qubits*, starting from very simple George Boole equation. We derive a generic and compact expression for basis vectors of *qubit* which can be useful in a further application. We also derive a generic form for the projection operator in the quantum information space. The results are also extended to higher dimensional d-level cases of *qutrits* and *qudits*.

1 Introduction

Quantum information theory is related to the most fundamental aspects of the computer science. In this work we investigate the transition from classical to quantum information. We propose a framework for understanding the relationship between *bit* and *qubit* based on Boole equation $x^2 = x$. The same procedure is then applied for devising a general expression for the basis vectors of the d-level quantum information unit *qudit*, as in (17). In fact, we demonstrate that the elements of orthonormal basis in d-dimensional Hilbert space C^d can be represented in a very simple generic form.

2 Bit versus Qubit

The elementary unit of information in classical computation is the *Shannon bit* or simply *bit*, which can take only two values $x \in \{0, 1\}$. Shannon bit was introduced under straightforward influence of the ideas of the great mathematician

and thinker George Boole exposed in his celebrated book [1]. Here, on the page 22 Boole introduced his famous equation

$$x^2 = x \tag{1}$$

and continued on the page 26:

> "We have seen ... that the symbols of Logic are subjects to the special law $x^2 = x$. Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law . We know that $0^2 = 0$ and that $1^2 = 1$; and the equation, considered as algebraic, has no other roots than 0 and 1."

We are going to demonstrate that qubit, a quantum generalization of bit, could be deduced from the particular matrix generalization of the same equation, namely

$$P(x)^2 = P(x), \quad x \in \{0,1\} \tag{2}$$

where x is the solution of the usual Boole equation (1). Equation (2)with normalization condition $TrP(x) = 1$ can be solved analytically to give

$$P(x) = \begin{pmatrix} 1-x & 0 \\ 0 & x \end{pmatrix} \tag{3}$$

It is straightforward to see this using identities $x^2 = x$ and $(1-x)^2 = 1-x$, which holds for any $x \in \{0,1\}$. The solution (2) corresponds to the *projection operator* (*state operator or filter operator*) in Quantum Mechanics [2], [3] and in general any projection operator P has the property $P^2 = P$. Projection operator *P(x)* from (3) can be represented in terms of Dirac's *kets*

$$\mid x >= \begin{pmatrix} 1-x \\ x \end{pmatrix} \tag{4}$$

and *bras*

$$< x \mid= \begin{pmatrix} 1-x & x \end{pmatrix} \tag{5}$$

as outer $ket \otimes bra$ product

$$P(x) =\mid x >< x \mid \tag{6}$$

From the definition (4) it follows the familiar form of two basis vectors $\mid 0 >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$and $\mid 1 >= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which confirms that Dirac's *kets* $\mid 0 >$and $\mid 1 >$ are quantum generalizations of Boolean 0 and 1.

3 Extension to Qutrit and Qudit

3.1 Qutrit

The amount of information in 3-level (ternary) classical system is named *trit* and can assume three values, such as $x \in \{yes, no, unknown\}$ or $x \in \{0, 1, 2\}$. Similarly the unit of quantum information in 3-level quantum system (e. g. spin 1 particle in magnetic field) is called *quantum trit* or *qutrit.* We shall show that the appropriate 3×3 solution of matrix generalization of the Boole equation can be used to introduce 3-dimensional normalized *qutrit* ket vector in compact form

$$\mid x >= \frac{1}{2} \begin{pmatrix} (1-x)(2-x) \\ 2x(2-x) \\ x(x-1) \end{pmatrix}, \quad x \in \{0,1,2\} \tag{7}$$

The classical Boole equation for a *trit* is a cubic equation

$$x(x-1)(x-2) = 0 \tag{8}$$

The corresponding quantum matrix equation

$$P(x)^2 = P(x), \quad x \in \{0,1,2\} \tag{9}$$

has 3×3 matrix solution in the form

$$P(x) = \frac{1}{2} \begin{pmatrix} (1-x)(2-x) & 0 & 0 \\ 0 & 2x(2-x) & 0 \\ 0 & 0 & x(x-1) \end{pmatrix} \tag{10}$$

which can be also expressed as outer product $P(x) =\mid x >< x \mid$ with $\mid x >$ given as in (7). From (7)and (10) we can see that qutrit is described by the following three basis vectors

$$\mid 0 >= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mid 1 >= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mid 2 >= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{11}$$

with corresponding projection operators

$$P(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, P(1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, P(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{12}$$

3.2 Qudit

A unit of quantum information in d-level quantum system, *qudit,* can be introduced in the same manner. For simplicity, let us consider first the particular

case for d=4. The same considerations allow to obtain the basis *ket* vectors of 4-level system as

$$| x >= \frac{1}{6} \begin{pmatrix} (1-x)(2-x)(3-x) \\ 3x(2-x)(3-x) \\ 3x(x-1)(3-x) \\ x(x-1)(2-x) \end{pmatrix} \quad x \in \{0,1,2,3\} \tag{13}$$

and the corresponding 4×4 projection operator *P(x)* which has only non-null diagonal terms defined by entries of the vector above, that is

$$P(x) = \frac{1}{6}\, diag((1-x)(2-x)(3-x),\, 3x(2-x)(3-x),\, 3x(x-1)(3-x),\, x(x-1)(2-x)) \tag{14}$$

This projection operator is a solution of matrix Boole equation for $x \in \{0,1,2,3\}$ with properties

$$P(x)^2 = P(x), \quad TrP(x) = 1 \tag{15}$$

The completeness relation is fulfilled:

$$\sum_{x=0}^{3} P(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

It is easy to generalize this results to the case of general *qudit*. For d-level system $x \in \{0,1,2,...,d-1\}$ the basis ket vector takes form

$$| x >= \begin{pmatrix} \frac{(1-x)(2-x)\cdots\cdots\cdots(d-1-x)}{(d-1)!} \\ \frac{(0-x)(2-x)\cdots\cdots\cdots(d-1-x)}{(d-2)!} \\ \vdots \\ \frac{1}{(-1)^k k!((d-1-k)!)} \prod_{k'=0,\, k' \neq k}^{d-1} (k' - x) \\ \vdots \\ \frac{(0-x)(1-x)\cdots\cdots\cdots(d-2-x)}{(-1)^{d-1}(d-1)!} \end{pmatrix} \quad x \in \{0,1,2,...,d-1\} \tag{17}$$

A general normalized d-dimensional vector can be expanded in this basis as

$$\sum_{x=0}^{d-1} a_x \mid x > \tag{18}$$

where a_x are complex numbers satisfying $\sum_x |a_x|^2 = 1$.

4 Example

The representation of qubit basis vectors in the form (4) allows to represent the entangled Bell basis for two qubits in a compact form. Using well known circuit

constructed from *Hadamard* and *cnot* gates, we obtain from (4) the following explicit compact form for Bell states

$$|B_{xy}>=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} (1-x)(1-y) \\ y-xy \\ x-xy \\ xy \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1-y \\ y \\ y-2xy \\ (1-2x)(1-y) \end{pmatrix} \tag{19}$$

which can be compared with other well known expressions, e. g. in [4].

5 Conclusions

Starting from a rather simple Boole equation and its matrix generalization we have devised a generic and compact representation of basis vectors for *qubit, qutrit* and general *qudit* case. We hope that our results could help in formalizing quantum algorithms and utilized by the researcher in the field of quantum information and computing.

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References

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